$$V_1 = V_2 = {}^1/_2\mu + {}^1/_2(\lambda + 2\mu)\{(1 - u_{3,3})^2 + u_{1,3}^2\} \pm {}^1/_2 D^{1/2}$$
$$D = \{\mu - (\lambda + 2\mu)(1 - u_{3,3})^2\}^2 + (\lambda + 2\mu)\{2(\lambda + 2\mu)(1 - u_{3,3})^2 + 2\mu + (\lambda + 2\mu)u_{1,3}^2\}u_{1,3}^2, \quad V_3 = \mu \qquad (2.13)$$

The propagation velocities of the sound waves are similar to the velocities of weak shockwaves. Therefore, weak shockwaves, as well as sound waves, in a three-dimensional elastic medium can propagate at three different velocities, one of which equals the transverse wave velocity in the linear approximation.

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ASYMPTOTIC ANALYSIS OF WAVE MOTIONS OF A VISCOUS FLUID WITH A FREE BOUNDARY

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Asymptotic expansions for the solution of the Cauchy-Poisson problem of wave motion of a viscous incompressible fluid with infinite depth are constructed at large Reynolds numbers. A proof of the asymptotics is given. Examples of plane and spatial motions are presented in which the asymptotic expansion is determined in the form of a free surface.

In the case of plane motion a solution of this problem was obtained in closed form and was analyzed in some particular cases in [1] by the integral transformation method. The problem was solved by the same method in other papers also. A discussion of these papers is presented in [2].

Moiseev proposed the asymptotic method [3-7] for the solution of this and a number of other problems.

Theorems of existence and uniqueness for solutions of unsteady linearized Navier-Stokes equations for the motion of a viscous fluid with a free surface in an open vessel were obtained in papers [8-10] in the absence and presence of surface tension.

In this paper an asymptotic method is also proposed. However, the method used for finding the asymptotics leads to simpler and more convenient expressions for numerical analysis than in [1, 3].

In Sect. 2 asymptotic expansions of the solution at large Reynolds numbers are constructed with any arbitrary preassigned degree of accuracy. The construction of the asymptotics is carried out by the method presented in paper [11]. In this connection the first and second iteration processes are applied simultaneously to the equations and boundary conditions. As a result of this, the initial system at each stage decomposes into two independent problems for the potential and vortical parts of the motion.

In Sect. 3 a proof of the method is given and an estimate is made of errors of asymptotic expansions in spaces with an energy norm.

In Sect. 4 examples of plane and spatial fluid motion are examined which arise as a result of normal surface tension and initial elevation of free surface. The first several terms of the asymptotic expansion of the elevation of the free surface are found.

1. Formulation of the problem. The Cauchy-Poisson problem for linearized Navier-Stokes equations of motion of a viscous incompressible fluid in a half-space with a free boundary is examined

$$\partial \mathbf{v}/\partial t = -\nabla p + \varepsilon^2 \Delta \mathbf{v}, \quad \operatorname{div} \mathbf{v} = 0, \quad p = p_r + \rho g z \tag{1.1}$$

$$\mathbf{v} = \mathbf{a}, \quad \operatorname{div} \mathbf{a} = 0, \quad \zeta = \zeta_*(x, y) \quad \text{for} \quad t = 0 \tag{1.2}$$

$$-p + \lambda\zeta + 2\varepsilon^2 \partial v_z/\partial z = -p_*(x, y, t), \quad \partial\zeta/\partial t = v_z \quad \text{for} \quad z = 0 \tag{1.3}$$

$$\varepsilon^2\left(\frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x}\right) = T_1(x, y, t), \quad \varepsilon^2\left(\frac{\partial v_y}{\partial z} + \frac{\partial v_z}{\partial y}\right) = T_2(x, y, t)$$

$$\lim_{r\to\infty} \mathbf{F} r^{1+\delta} = 0, \quad \delta \geqslant {}^1/_2 \quad (\text{in the case of plane motion } \delta \geqslant -{}^1/_2)$$

$$\mathbf{F} \equiv \{\mathbf{v}, \partial\mathbf{v}/\partial x, \partial\mathbf{v}/\partial y, \partial\mathbf{v}/\partial z, p, \zeta_*, T_1, T_2, p_*, \mathbf{a}\}, \; r = (x^2 + y^2 + z^2)^{1/2} \tag{1.4}$$

The quantities in (1.1)–(1.4) are dimensionless. They are related to dimensional quantities (which are designated by a prime) by the following relationships:

$$x' = \alpha x, \quad y' = \alpha y, \quad z' = \alpha z, \quad \zeta' = \alpha\zeta, \quad \zeta_*' = \alpha\zeta_*, \quad t' = \beta t$$

$$\mathbf{a}' = \frac{\alpha}{\beta}\mathbf{a}, \quad \mathbf{v}' = \frac{\alpha}{\beta}\mathbf{v}, \quad p' = \rho\frac{\alpha^2}{\beta^2}p, \; p_r' = \rho\frac{\alpha^2}{\beta^2}p_r, \quad p_*' = \rho\frac{\alpha^2}{\beta^2}p_*$$

$$T_{1,2}' = \rho\frac{\alpha^2}{\beta^2}T_{1,2} \quad \varepsilon^2 = \frac{1}{R} \quad R = \frac{\alpha^2}{\nu\beta}, \quad \lambda = \frac{1}{F}, \quad F = \frac{\alpha}{g\beta^2}$$

Here $\mathbf{v}'$ is the velocity vector, p_r' is the hydrodynamic pressure, ζ' is the elevation of the free surface, α is the length unit β is the unit of time, ν is the kinematic coefficient of viscosity, ρ is the density of the fluid, g is the gravitational acceleration, R is the Reynolds number and F is the Froude number. The origin of coordinates is taken on the unperturbed surface. The axis Oz is pointed vertically up. The fluid is set in motion by initial elevation of the free surface ζ_*' by external surface tension $\mathbf{p}_n' \equiv \equiv \{p_*', T_1', T_2'\}$ and by initial potential field of velocities $\mathbf{a}'$.

2. Construction of the asymptotics. Asymptotic expansions of the solution of the problem (1.1)–(1.4) for $\varepsilon \to 0$ are constructed in the following form:

$$\mathbf{v} = \sum_{i=0}^{k} \varepsilon^i \mathbf{v}_i + \sum_{i=-1}^{k} \varepsilon^i \mathbf{g}_i + \mathbf{u}, \qquad p = \sum_{i=0}^{k} \varepsilon^i p_i + \sum_{i=0}^{k} \varepsilon^i h_i + q$$

$$\zeta = \sum_{i=0}^{k} \varepsilon^i \zeta_i + \sum_{i=1}^{k} \varepsilon^{i+1}\theta_i + \eta, \quad \frac{\partial\zeta_i}{\partial t} = v_{iz}, \qquad \frac{\partial\theta_i}{\partial t} = g_{iz} \tag{2.1}$$

$$\partial\eta/\partial t = u_z, \quad z = 0, \quad \zeta_0 = \zeta_*, \; \zeta_{i+2} = \theta_i' = \eta = 0, \; t = 0 \quad (i = -1, 0, 1 \ldots)$$

Here
$$\mathbf{v}_i = \mathbf{v}_i(x, y, z, t), \; p_i = p_i(x, y, z, t), \; \zeta_i = \zeta_i(x, y, t)$$

are obtained as a result of the first iteration process [11]. That is, denoting the left part by $\mathbf{P}$, we require that

$$\mathbf{P}(\mathbf{V}_k) = O(\varepsilon^k), \qquad \mathbf{V}_k = \left\{ \sum_{i=0}^{k} \varepsilon^i \mathbf{v}_i \quad \sum_{i=0}^{k} \varepsilon^i p_i \right\}$$

The coefficients of $\varepsilon^0, \varepsilon^1, \varepsilon^2, \ldots \varepsilon^k$ are subsequently set equal to zero and the following system is obtained for the determination of $\mathbf{v}_0, p_0, \zeta_0$:

$$\begin{gathered} \partial \mathbf{v}_0/\partial t = -\nabla p_0 \qquad \operatorname{div} \mathbf{v}_0 = 0 \\ \mathbf{v}_0 = \mathbf{a},\ \zeta_0 = \zeta_*,\ t = 0;\ \mathbf{v}_0,\ \partial \mathbf{v}_0/\partial x,\ \partial \mathbf{v}_0/dy \to 0,\ x^2 + y^2 \to \infty \qquad (2.2) \\ \partial \zeta_0/\partial t = v_{0z},\ -p_0 + \lambda\zeta_0 = -p_* - \lambda\theta_{-1},\ z = 0;\ \mathbf{v}_0 \to 0,\ z \to -\infty \end{gathered}$$

For the determination of $\mathbf{v}_i, p_i$ $(i \geqslant 1)$ we obtain the system

$$\begin{gathered} \partial \mathbf{v}_i/\partial t = -\nabla p_i + \Delta \mathbf{v}_{i-2},\ \operatorname{div} \mathbf{v}_i = 0;\ \mathbf{v}_i = 0,\ \zeta_i = 0,\ t = 0;\ \mathbf{v}_i,\ \partial \mathbf{v}_i/\partial x \\ \partial \mathbf{v}_i/\partial y \to 0,\ x^2 + y^2 \to \infty \quad (\mathbf{v}_{-1} \equiv 0) \qquad (2.3) \\ \partial \zeta_i/\partial t = v_{iz},\ -p_i + \lambda\zeta_i = -2\partial v_{i-2,z}/\partial z - \lambda\theta_{i-1} - 2\partial g_{i-2,z}/\partial s \\ z = 0; \qquad v_i \to 0, \qquad z \to -\infty \end{gathered}$$

The function θ_{-1} in (2.2) represents the displacement thickness which is well known in the boundary layer theory [12].

The expressions $\Delta \mathbf{v}_{i-2}$ $(i \geqslant 2)$ are equal to zero. For $\Delta \mathbf{v}_0$ this follows from the condition of vector $\mathbf{a}$ being a potential vector and the first two equations in (2.2). For the remaining expressions this follows from the first three equations in (2.3).

Vectors $\mathbf{g}_i$ $(x, y, z, t, \varepsilon) = \{g_{ix}, g_{iy}, \varepsilon g_{iz}\}$ and functions θ_i (x, y, t, ε) and h_i $(x, y, z, t, \varepsilon)$ are found using the second iteration process [11] and they compensate for discrepancies of $\mathbf{v}_i, \zeta_i, h_i$ in satisfying boundary conditions (1.3). The solution is sought in the form

$$\begin{gathered} \mathbf{v} \sim \sum_{i=0}^{k} \varepsilon^i \mathbf{v}_i + \sum_{i=-1}^{k} \varepsilon^i \mathbf{g}_i, \qquad p \sim \sum_{i=0}^{k} \varepsilon^i p_i + \sum_{i=-1}^{k} \varepsilon^i h_i \\ \zeta \sim \sum_{i=0}^{k} \varepsilon^i \zeta_i + \sum_{i=-1}^{k} \varepsilon^{i+1}\theta_i, \qquad \mathbf{g}_i = \{g_{ix}, g_{iy}, \varepsilon g_{iz}\} \qquad (2.4) \end{gathered}$$

We substitute (2.4) in (1.1)–(1.4), take into account (2.2) and (2.3) and subsequently assume $z = \varepsilon s$ and set the coefficients of $\varepsilon^{-1}, \varepsilon^0, \ldots, \varepsilon^k$ equal to zero. As a result we obtain for the determination of g_i and h_i the following set of equations:

$$\begin{gathered} Lg_{ix} = -\frac{\partial h_i}{\partial x}, \quad Lg_{iy} = -\frac{\partial h_i}{\partial y}, \quad \frac{\partial g_{iz}}{\partial s} = -\frac{\partial g_{ix}}{\partial x} - \frac{\partial g_{iy}}{\partial y} \\ Lf_i \equiv \frac{\partial f_i}{\partial t} - \frac{\partial^2 f_i}{\partial s^2} - \Delta_1 f_{i-2}, \quad \Delta_1 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \quad (i = -1, 0, \ldots k) \\ \frac{\partial h_{m+2}}{\partial s} = -Lg_{mz} \quad (m = -3, -2, \ldots k) \qquad (g_{-5}, g_{-4}, g_{-3}, g_{-2} = 0) \qquad (2.5) \end{gathered}$$

with boundary conditions at infinity

$$g \to 0, \quad h_i \to 0; \quad s \to -\infty \qquad (2.6)$$

These boundary conditions result from the requirement that the vectors g_i and functions h_i must have the character of a boundary layer. We can show that all $h_m \equiv 0$.

For h_{-1}, h_0 this follows directly from the last equation in (2.5) and (2.6). For the remaining h_m we use the method of mathematical induction. Let us assume that $h_m = 0$.

Then we will show that $h_{m+2} \equiv 0$. We differentiate the last equation in (2.5) with respect to s and use the third equation for $i = m$. By virtue of the first two equations in (2.5) and also (2.6) we derive that $h_{m+2} \equiv 0$. Now for determination of g_i and θ_i we derive from (2.5)

$$Lg_{ix} = 0, \quad Lg_{iy} = 0, \quad \frac{\partial g_{iz}}{\partial s} = -\frac{\partial g_{ix}}{\partial x} - \frac{\partial g_{iy}}{\partial y}$$

$$Lf_i \equiv \frac{\partial f_i}{\partial t} - \frac{\partial^2 f_i}{\partial s^2} - \Delta_1 f_{i-2}, \quad \frac{\partial \theta_i}{\partial t} = g_{iz}, \; s = 0$$

$$\theta_i = 0, t = 0 \quad (i = -1, 0, 1, \ldots k) \quad (g_{-3}, g_{-2} = 0) \tag{2.7}$$

with initial boundary conditions

$$g_i = 0, \quad t = 0, \quad g_i \to 0, \quad s \to -\infty \quad (i = -1, 0, 1, \ldots k) \tag{2.8}$$

$$\frac{\partial g_{-1x}}{\partial s} = T_1(x, y, t), \quad \frac{\partial g_{-1y}}{\partial s} = T_2(x, y, t), \quad s = 0 \tag{2.9}$$

$$\frac{\partial g_{ix}}{\partial s} = -\frac{\partial v_{i-1x}}{\partial z} - \frac{\partial v_{i-1z}}{\partial x} - \frac{\partial g_{i-2z}}{\partial x} \equiv A_{1i} \quad (i = 0, 1, \ldots k)$$

$$(s = 0) \tag{2.10}$$

$$\frac{\partial g_{iy}}{\partial s} = -\frac{\partial v_{i-1y}}{\partial z} - \frac{\partial v_{i-1z}}{\partial y} - \frac{\partial g_{i-2z}}{\partial y} \equiv A_{2i} \; (v_{-1} = g_{-2} = 0)$$

The boundary value problems (2.2), (2.3) and (2.7)-(2.10) are solved through the application of integral Fourier transformations with respect to coordinates x and y and Laplace's transformation with respect to time t [13]. In particular, for the first five terms of the asymptotic expansion for the elevation of the free surface we have

$$\xi = \xi_0 + \varepsilon\xi_1 + \cdots + \varepsilon^4\xi_4, \quad \xi_i = \zeta_i + \theta_{i-1} \tag{2.11}$$

$$L\Phi\xi = L\Phi\xi_0 + \varepsilon L\Phi\xi_1 + \cdots + \varepsilon^4 L\Phi\xi_4, \quad L\Phi\xi_i = L\Phi\zeta_i + L\Phi\theta_{i-1}$$

$$L\Phi\xi = \Phi\zeta_* \left[\frac{\sigma}{\sigma^2 + a\lambda} + \frac{\lambda}{\sigma}\chi\right] + L\Phi p_* \left[-\frac{a}{\sigma^2 + a\lambda} + \chi\right] + \Phi a_{*z}\left[\frac{1}{\sigma^2 + a\lambda} - \frac{1}{a}\chi\right] -$$

$$- L\Phi\left(\frac{\partial T_1}{\partial x} + \frac{\partial T_2}{\partial y}\right)\left\{\frac{1}{(\sigma^2 + a\lambda)} - \varepsilon\frac{2a}{\sqrt{\sigma}(\sigma^2 + a\lambda)} + \varepsilon^2\left[\frac{2a^2}{\sigma(\sigma^2 + a\lambda)} - \frac{4a^3\sigma}{(\sigma^2 + a\lambda)^2}\right] + \right.$$

$$\left. + \varepsilon^3\left[\frac{12a^3\sqrt{\sigma}}{(\sigma^2 + a\lambda)^2} - \frac{a^3}{\sigma^{3/2}(\sigma^2 + a\lambda)}\right] + \varepsilon^4\left[\frac{4a^4(3\sigma^2 - a\lambda)}{(\sigma^2 + a\lambda)^3} - \frac{16a^4}{(\sigma^2 + a\lambda)^2}\right]\right\}$$

$$\chi = \varepsilon^2\frac{4a^3\sigma}{(\sigma^2 + a\lambda)^2} - \varepsilon^3\frac{4a^4\sqrt{\sigma}}{(\sigma^2 + a\lambda)^2} - \varepsilon^4\frac{4a^5(3\sigma^2 - a\lambda)}{(\sigma^2 + a\lambda)^3} \tag{2.12}$$

$$L\Phi f = \int_0^\infty \Phi f e^{-\sigma t} dt, \quad \Phi f = \frac{1}{2\pi}\iint_{-\infty}^{\infty} f(x, y, t) e^{i(\xi x + \eta y)} dx dy, \quad a = \sqrt{\xi^2 + \eta^2}$$

The inverse transforms of L-representations for the function $\Phi\xi$ are found by using the convolution theorem and tables [14]. The integral representation of the function ξ is obtained by applying the inversion formula for the Fourier transformation.

We note that Eqs. (2.12) can be obtained by another method [15]. For this purpose the method of integral transformations is applied directly to (1.1)–(1.4) and the transform $L\Phi\xi$ is expanded in a series with respect to ε.

Note 2.1. In the absence of shear stresses the function θ_{-1} is equal to zero and for the determination of v_0, p_0, ζ_0 we obtain a known problem of irrotational motion [16].

Note 2.2. If the shear stresses are not equal to zero on the free surface, then $g_{-1} \neq 0$ and the components $\varepsilon^{-1} g_{-1x}$ and $\varepsilon^{-1} g_{-1y}$ of the velocity vector increase without bounds on the free surface $z = 0$ when $\varepsilon \to 0$. The physical explanation for this is that a perfect fluid does not perceive shear forces. At the same time the corresponding contribution of the boundary layer to the elevation of the free surface remains finite (see (2.12)). Asymptotic expansions for the form of the free surface have nonzero coefficients for all powers of ε which are positive integers. In the absence of shear stresses the coefficient for ε is equal to zero, however, the coefficients for ε^2, ε^3, etc. are different from zero (see (2.12)).

Note 2.3. The method for construction of the asymptotics is described for the spatial problem. It is apparent that all arguments are analogous for the plane case. In order to obtain the corresponding formulas, it is necessary to set equal to zero all derivatives with respect to y, and also to assume that $v_y, v_{iy}, g_{iy} = 0$ $(i = -1, 0, \ldots k)$.

Note 2.4. The described method for the construction of asymptotic expansions can be applied to the case of wave motions of a fluid in a layer or in a vessel of arbitrary shape.

3. Proof of asymptotic expansions (3.1). Let us introduce Banach spaces $L_2(E)$ of functions $f(\omega)$, $\omega = \{x, y, z\}$ defined in the half-space E $(z \leqslant 0)$ and $L_2(\Gamma)$ of functions $\varphi(\gamma)$, $\gamma = \{x, y, 0\}$ defined in the plane Γ $(z = 0)$ with a finite norm

$$L_2(E) \quad \|f\|_E^2 = \int_E f^2 d\omega, \qquad L_2(\Gamma) \quad \|\varphi\|_\Gamma^2 = \int_\Gamma \varphi^2 d\gamma \tag{3.1}$$

Let us further define the Banach space H of vector functions $\mathbf{u} = \{u_x, u_y, u_z\}$ with a finite norm

$$(H) \quad \|\mathbf{u}\|_H^2 = \|u_x\|^2 + \|u_y\|^2 + \|u_z\|^2 \tag{3.2}$$

We also introduce the Banach space H_1 of vector functions $\mathbf{u}$ which vanish at infinity and which have first generalized derivatives summable with a square over the half-space E with a finite norm

$$\|\mathbf{u}\|_{H_1}^2 = \int_E (|\nabla u_x|^2 + |\nabla u_y|^2 + |\nabla u_z|^2)\, d\omega \tag{3.3}$$

Following [9], we introduce the notation

$$\mathbf{E}(\mathbf{u}, \mathbf{v}) = \int_E \Big[2 \frac{\partial u_x}{\partial x} \frac{\partial v_x}{\partial x} + 2 \frac{\partial u_y}{\partial y} \frac{\partial v_y}{\partial y} + 2 \frac{\partial u_z}{\partial z} \frac{\partial v_z}{\partial z} + \left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) + \\ + \left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right) + \left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right)\left(\frac{\partial v_y}{\partial z} + \frac{\partial v_z}{\partial y}\right) + \left(\frac{\partial u_z}{\partial x} + \frac{\partial u_x}{\partial z}\right) + \\ + \left(\frac{\partial v_z}{\partial x} + \frac{\partial v_x}{\partial z}\right)\Big] d\omega \tag{3.4}$$

For vector functions $\mathbf{u} \in H_1$ Korn's inequality is applicable [9, 17]

$$\|\mathbf{u}\|_{H_1}^2 \leqslant c\mathbf{E}(\mathbf{u}, \mathbf{u}) \tag{3.5}$$

Here c is some positive constant. Now Green's formula which is valid for solenoidal vectors $\mathbf{v}$ and $\mathbf{u}$ will be applied to Navier-Stokes equations

$$\int_E (-\varepsilon^2 \Delta \mathbf{u} + \nabla p)\, \mathbf{v} d\omega = \varepsilon^2 \mathbf{E}(\mathbf{u}, \mathbf{v}) - \int_\Gamma \Big[\varepsilon^2 \left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right) v_x + \\ + \varepsilon^2 \left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right) v_y + \left(-p + 2\varepsilon^2 \frac{\partial u_z}{\partial z}\right) v_z\Big] d\gamma \tag{3.6}$$

The values $\mathbf{u}$, q, η from (2.1) are substituted into the left side of system (1.1)–(1.4). Using (2.2), (2.3) and (2.7)–(2.10), we obtain

$$\partial \mathbf{u}/\partial t + \nabla q - \varepsilon^2 \Delta \mathbf{u} = \varepsilon^{k+1}\mathbf{f}, \qquad \operatorname{div}\mathbf{u} = 0$$

$$\eta = 0, \quad \mathbf{u} = 0, \quad t = 0; \qquad \mathbf{u},\ \partial\mathbf{u}/\partial x,\ \partial\mathbf{u}/\partial y \to 0, \quad x^2 + y^2 \to \infty$$

$$\varepsilon^2\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right) = \varepsilon^{k+2}\varphi_1, \quad \varepsilon^2\left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right) = \varepsilon^{k+2}\varphi_2, \quad \frac{\partial \eta}{\partial t} = u_z$$

$$-q + \lambda\eta + 2\varepsilon^2\frac{\partial u_z}{\partial z} = \varepsilon^{k+1}\varphi_3, \quad z = 0; \qquad \mathbf{u} \to 0; \quad z \to -\infty \tag{3.7}$$

Here

$$\mathbf{f} \equiv \{f_x, f_y, f_z\}, \quad -f_x = \Delta_1 g_{k-1x} + \varepsilon\Delta_1 g_{kx}, \quad -f_y = \Delta_1 g_{k-1y} + \varepsilon\Delta_1 g_{ky}$$

$$f_z = -\varepsilon(\Delta_1 g_{k-1z} + \varepsilon\Delta_1 g_{kz}), \qquad \Delta_1 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \tag{3.8}$$

$$\varphi_1(x, y, t) = \left[\varepsilon\frac{\partial g_{kz}}{\partial x} - \frac{\partial g_{k+1x}}{\partial s}\right]_{z=0}, \quad \varphi_2(x, y, t) = \left[\varepsilon\frac{\partial g_{kz}}{\partial y} - \frac{\partial g_{k+1y}}{\partial s}\right]_{z=0}$$

$$\varphi_3(x, y, t) = \lambda\theta_k + 2[R_{k-1} + \varepsilon R_k]_{z=0}, \quad R_k = \frac{\partial v_{kz}}{\partial z} + \frac{\partial g_{kz}}{\partial s}$$

Theorem 3.1. Let the vector function $\mathbf{f}$ and any of its derivatives with respect to x and y up to the jth order (j being a sufficiently large number) belong to the space H, and let functions $\varphi_1, \varphi_2, \varphi_3$ together with their derivatives to the jth order belong to the space $L_2(\Gamma)$. Then for the solution of problem (3.7), (3.8) the following estimates are valid:

$$\|D^l\mathbf{u}\|_H \leqslant C_l\varepsilon^k, \quad \|D^l\eta\|_\Gamma \leqslant C_l\lambda^{-1/2}\varepsilon^k \quad (l = 0, 1, \ldots j)$$

$$\max_\Gamma |D^\alpha\eta| \leqslant M_\alpha\varepsilon^k \qquad (\alpha = 0, 1, 2, \ldots l-2) \tag{3.9}$$

$$C_l = \left(c\int_0^t m_l^2(\tau)\,d\tau\right)^{1/2} + 2\varepsilon\int_0^t n_l(\tau)\,d\tau, \quad n_l = m_l + \|D^l\mathbf{f}\|_H$$

$$m_l = \sqrt{3}\max\{\varepsilon\|D^l\varphi_1\|_\Gamma, \varepsilon\|D^l\varphi_2\|_\Gamma, \|\varphi_3\|_\Gamma\}$$

(Here and everywhere in the subsequent text D^l_φ represents an arbitrary mixed derivative of the function $\varphi(x, y, z)$ with respect to x and y and of the order l.)

Proof. First, let us note a simple inequality which is valid for functions w such that $w \to 0$ for $z \to -\infty$

$$\|w\|_\Gamma \leqslant \|w\|_E + \left\|\frac{\partial w}{\partial z}\right\|_E \tag{3.10}$$

The validity of this inequality follows from relationships

$$\int_\Gamma w^2 d\gamma = 2\int_\Gamma\int_{-\infty}^0 w\frac{\partial w}{\partial z}\,dz\,d\gamma = 2\int_E w\frac{\partial w}{\partial z}\,d\omega \leqslant 2\|w\|_E\left\|\frac{\partial w}{\partial z}\right\|_E \leqslant \|w\|_E^2 + \left\|\frac{\partial w}{\partial z}\right\|_E^2$$

Now the first equation (3.7) is multiplied by $\mathbf{u}$. We integrate over the domain E taking into account (3.7) and Green's formula. In Green's formula we preliminarily set $\mathbf{v} = \mathbf{u}$. We obtain

$$\frac{1}{2}\frac{d}{dt}[\|\mathbf{u}\|_H^2 + \lambda\|\eta\|_\Gamma^2] + \varepsilon^2 E(\mathbf{u}, \mathbf{u}) =$$

$$= \varepsilon^{k+1}\int_E \mathbf{fu}\,d\omega + \varepsilon^{k+1}\int_\Gamma[\varepsilon(\varphi_1 u_x + \varphi_2 u_y) + \varphi_3 u_z]\,d\gamma \tag{3.11}$$

Applying to the left side of (3.11) Korn's inequality (3.5) and to the right side at first the Cauchy-Buniakowski inequality, then (3.10), and also the following obvious inequalities

$$\varepsilon^{k+1} m_0 \left\| \frac{\partial \mathbf{u}}{\partial z} \right\|_{\mathrm{H}} \leqslant \varepsilon^{k+1} m_0 \|\mathbf{u}\|_{\mathrm{H}_1} \leqslant \frac{c}{2} \varepsilon^{2k} m_0^2 + \frac{\varepsilon^2}{2c} \|\mathbf{u}\|_{\mathrm{H}_1}^2$$

we derive

$$\frac{1}{2}\frac{d}{dt}\left[\|\mathbf{u}\|_{\mathrm{H}}^2 + \lambda \|\eta\|_{\Gamma}^2\right] + \frac{\varepsilon^2}{2c}\|\mathbf{u}\|_{\mathrm{H}_1}^2 \leqslant \varepsilon^{k+1} n_0 \|\mathbf{u}\|_{\mathrm{H}} + \frac{c}{2}\varepsilon^{2k} m_0^2 \tag{3.12}$$

$$n_0 = m_0 + \|\mathbf{f}\|, \qquad m_0 = \sqrt{3}\max\{\varepsilon\|\varphi_1\|_{\Gamma}, \quad \varepsilon\|\varphi_2\|_{\Gamma}, \quad \|\varphi_3\|_{\Gamma}\}$$

Utilizing the inequality

$$\|\mathbf{u}\|_{\mathrm{H}} + \sqrt{\lambda}\|\eta\|_{\Gamma} \leqslant \sqrt{2}\, Z, \qquad Z^2(t) = \|\mathbf{u}\|_{\mathrm{H}}^2 + \lambda\|\eta\|_{\Gamma}^2$$

we derive from (3.12) using (3.7)

$$dZ^2/dt \leqslant 4 n_0 \varepsilon^{k+1} Z + c\varepsilon^{2k} m_0^2, \qquad Z(0) = 0$$

From this by means of the method proposed in [19] (pp. 565, 566), we obtain

$$Z(t) \leqslant 2\varepsilon^{k+1}\int_0^t n_0(\tau)\, d\tau + \varepsilon^k c^{1/2}\left\{\int_0^t m_0^2(\tau)\, d\tau\right\}^{1/2} \tag{3.13}$$

Then for $l = 0$ the first two estimates in (3.9) follow from (3.13). In order to obtain estimates (3.9) for the derivatives, it is necessary to differentiate Eqs. (3.7) and (3.8) with respect to x and y a corresponding number of times and to repeat exactly the same arguments as were used in the derivation of (3.13). It is not difficult to see that in the formulas now instead of φ_1, φ_2, φ_3 and $\mathbf{f}$ their derivatives of the same order as in the left side will be present. The third estimate follows from the second one with the aid of simple inequalities. For example, in the case of the plane problem for $\alpha = 0$ we have

$$\max_x |\eta(x, t)| \leqslant \left(2\int_{-\infty}^{x} \eta \frac{\partial \eta}{\partial x} dx\right)^{1/2} \leqslant \left[2\int_{-\infty}^{\infty} \eta^2 dx \int_{-\infty}^{\infty}\left(\frac{\partial \eta}{\partial x}\right)^2 dx\right]^{1/2} \leqslant$$

$$\leqslant (2C_0C_1\lambda^{-1})^{1/2}\varepsilon^k = M_0\varepsilon^k \tag{3.13'}$$

We note that by virtue of (3.8) functions $C_l(t)$, $M_\alpha(t)$ in estimates (3.9) depend on functions which are determined as a result of iteration processes.

At any finite interval of time $0 < t < t_0$ we can obtain immediately estimates of the type (3.9) from the input data ζ_*, p_*, $\mathbf{a}$, T_1, T_2 for the problem. The input data have an exponential dependence on t_0.

Lemma 3.1. Let a constant N_1 exist such that

$$\|D^\alpha T_r\|_{\Gamma} \leqslant N_1 \; (r = 1, 2;\ 0 < t < t_0;\ \alpha = 0, 1, \ldots, j_1),\ t_0 > 1 \tag{3.14}$$

Then for the solution of the problem (2.7)–(2.9) for $i = -1$ the following estimates are valid in the interval $(0, t_0)$

$$\|D^\alpha \mathbf{g}_{-1}\|_{\mathrm{H}} \leqslant K_1\varepsilon^{1/2}, \ \{\|D^\alpha g_{-1z}\|_{\Gamma}, \ \left\|D^\alpha \frac{\partial g_{-1z}}{\partial s}\right\|_3, \ \|D^\alpha \theta_{-1}\|_{\Gamma}\} \leqslant K_2$$

$$\left\{\left\|D^\alpha \frac{\partial \mathbf{g}_{-1}}{\partial s}\right\|_{\mathrm{H}}, \left\|D^\alpha \left\|\frac{\partial^2 g_{-1z}}{\partial s^2}\right\|_E\right\|\right\} \leqslant K_3\varepsilon^{1/2} \tag{3.15}$$

Here the constants K_i are proportional to N_1.

Proof. From (2.7)–(2.9) we obtain

$$g_{-1x} = \int_0^t G(s, t-\tau)\, T_1(x, y, \tau)\, d\tau, \quad G(s, u) = \frac{-1}{\sqrt{\pi u}} e^{-s^2/4u} \tag{3.16}$$

From this we derive

$$\|D^\alpha g_{-1x}\|_E^2 \leqslant \varepsilon N_1^2 [I(-\infty,-1) + I(-1,0)] \leqslant \varepsilon K_1^2 t_0^{7/2}$$

$$I(a,b) = \int_a^b \left\{\int_0^t G(s, t-\tau)\, d\tau\right\}^2 ds \qquad (t_0>1)$$

We note that each of the intervals in square brackets is estimated separately. Further, from (2.7), (2.6), using (3.16) and a corresponding equation for g_{-1y}, we obtain

$$D^\alpha \frac{\partial g_{-1z}}{\partial s} = -\int_0^t G(s, t-\tau) D^\alpha \left(\frac{\partial T_1}{\partial x} + \frac{\partial T_2}{\partial y}\right) d\tau \tag{3.17}$$

From (3.17) using (3.14) we derive

$$\|D^\alpha g_{-1z}\|_E^2 \leqslant \varepsilon N_1^2 \int_{-\infty}^0 [I(-\infty, s)]^2 ds$$

Changing the order of integration in the square brackets, computing the inner integral and applying successively the following obvious inequalities:

$$1 - \Phi\left(\frac{s}{2\sqrt{u}}\right) \leqslant 1 - \Phi\left(\frac{s}{2\sqrt{t}}\right) \equiv \Phi_1, \qquad \Phi_1^2 \leqslant \Phi_1$$

$$\Phi(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-y^2} dy, \quad 0 < u < t, \quad -\infty < s \leqslant 0$$

and also formula 6.281 of [20], we arrive at the first estimate from (3.15). The remaining estimates are derived in an analogous manner.

Lemma 3.2. Let constants N_2 and N_3 exist such that (*)

$$\{\sqrt{2}(\|D^\alpha \mathbf{a}\|_H + \sqrt{\lambda}\|D^\alpha \zeta_*\|_\Gamma), \|D^\alpha \mathbf{a}\|_{H_1}\} \leqslant N_2 \quad (\alpha=0,1,\ldots j_2)$$

$$\|D^\alpha p_*\|_{W_2^{1/2}(\Gamma)} \leqslant N_3, \qquad \|D^\alpha \dot{\theta}_{-1}\|_{W_2^{1/2}(\Gamma)} \leqslant N_3 \tag{3.18}$$

Then for solution of problem (2.2) the following estimates hold:

$$\|D^\alpha \mathbf{v}_0\|_H \leqslant N_2 + K_3 t, \quad \left\|D^\alpha \frac{\partial \mathbf{v}_0}{\partial z}\right\|_H \leqslant K_4 + K_5 t$$

$$\left\|D^\alpha \frac{\partial^2 \mathbf{v}_0}{\partial z^2}\right\|_H \leqslant K_6 + K_7 t \qquad (\alpha=0,1,\ldots j-2) \tag{3.19}$$

$$\left\{\left\|D^\alpha \frac{\partial v_{0z}}{\partial z}\right\|_\Gamma, \left\|D^\alpha \frac{\partial v_{0x}}{\partial z}\right\|_\Gamma, \left\|D^\alpha \frac{\partial v_{0z}}{\partial x}\right\|_\Gamma\right\} \leqslant K_8 + K_9 t$$

Here the constants K_i do not depend on x, y, z, t, ε.

Proof. Let us differentiate the equations and boundary conditions in (2.2) with respect to x and y the appropriate number of times depending on which derivative is being estimated. The resulting vector equation is multiplied by $D^\alpha \mathbf{v}_0$ and integrated over the domain E taking into account the boundary conditions. Then we find

$$\frac{1}{2}\frac{d}{dt}[\|D^\alpha \mathbf{v}_0\|_H^2 + \lambda \|D^\alpha \zeta_0\|_\Gamma^2] = -\int_\Gamma D^\alpha v_{0z} D^\alpha (p_* + \lambda\theta_{-1})\, d\gamma \tag{3.20}$$

According to theorem (2.3) of [21] by virtue of (3.18), the function $p_* + \lambda\theta_{-1}$ can be

*) Definition of spaces W_2^l with nonintegral l (see Sect. 2 of [21]).

extended to the half-space E such that

$$\|\operatorname{grad} D^\alpha (p_* + \lambda\theta_{-1})\|_H \leqslant D^\alpha (p_* + \lambda\theta_{-1})\|_{W_2^1(E)} \leqslant c\, \|D^\alpha (p_* + \lambda\theta_{-1})\|_{W_2^{1/2}(\Gamma)} \tag{3.21}$$

Applying to the last integral in (3.20) Gauss' theorem, the Cauchy-Buniakowski inequality and (3.21), we obtain

$$\|D^\alpha v_0\|_H + \sqrt{\lambda}\, \|D^\alpha \zeta_0\|_\Gamma \leqslant \sqrt{2}\, (\|D^\alpha a\|_H + \sqrt{\lambda}\, \|D^\alpha \zeta_*\|_\Gamma) + c_1 \int_0^t \|D^\alpha (p_* + \lambda\theta_{-1})\|_{W_2^{1/2}(\Gamma)}\, d\tau$$

The first estimate in (3.19) follows then from here. (We note that θ_{-1} satisfies conditions (3.18) by virtue of theorem 2.3 of [21] and the third group of estimates in (3.15).) Let us now proceed to the proof of the second estimate in (3.18). From the divergence equation we derive directly

$$\left\| D^\alpha \frac{\partial v_{0z}}{\partial z} \right\|_E \leqslant \left\| D^\alpha \frac{\partial v_{0x}}{\partial x} \right\|_E + \left\| D^\alpha \frac{\partial v_{0y}}{\partial y} \right\|_E \tag{3.22}$$

Differentiating the first and second components of the vector equation in (2.2) with respect to z, the third component with respect to x or y, subtracting and integrating with respect to t, we obtain

$$\frac{\partial v_{0x}}{\partial z} - \frac{\partial v_{0z}}{\partial x} = \frac{\partial a_x}{\partial z} - \frac{\partial a_z}{\partial x}, \quad \frac{\partial v_{0y}}{\partial z} - \frac{\partial v_{0z}}{\partial y} = \frac{\partial a_y}{\partial z} - \frac{\partial a_z}{\partial y} \tag{3.23}$$

From here, using (3.22) and the first estimate in (3.19) we find the second estimate.

The estimate for the second derivative of v_0 with respect to z is derived from a chain of inequalities which are obtained by differentiating the continuity equations and (3.23) with respect to x, y, z, with subsequent utilization of the first two estimates from (3.19). The last estimate follows from the first three using (3.10).

Lemma 3.3. Let a constant N_3 exist such that for $0 < t < t_0$

$$\left\| D^\alpha \left(2 \frac{\partial v_{i-2z}}{\partial z} + \lambda\theta_{i-1} + 2 \frac{\partial g_{i-2z}}{\partial_s} \right\|_{W_2^{1/2}(\Gamma)} < N_3 \tag{3.24}$$

Then estimates (3.19) are valid for v_i. The proof of this lemma and Lemma 3.2 are identical word for word. It is only necessary to replace $v_0, p_0, \theta_{-1}, p_*$ by v_i, p_i, θ_{i-1}, $2\partial/\partial s\, (v_{i-2z} + g_{i-2z})$, respectively, and also to set $a = 0$, $\zeta_* = 0$.

Lemma 3.4. Let constants N_1, N_2 exist such that

$$\{\|D^\alpha A_{ri}\|_\Gamma, \|D^\alpha g_{i-2}\|_E\} \leqslant N_1 \qquad (r=1,2;\ 0<t<t_0) \tag{3.25}$$

Then the estimates (3.15) are valid for g_i, which are the solutions of problems (2.7)–(2.9) when $i \geqslant 0$ in the interval $(0, t_0)$.

Proof. Let us represent g_{ix} in the form of sum $g_{ix} = w_i + q_i$

$$Lw_i = 0, \quad \frac{\partial q_i}{\partial t} = \frac{\partial^2 q_i}{\partial s^2}; \qquad w_i = q_i = 0, \quad t = 0 \tag{3.26}$$

$$\frac{\partial w_i}{\partial s} = 0, \quad s=0; \quad \frac{\partial q_i}{\partial s} = A_{1i}; \qquad w_i, \quad q_i \to 0, \quad s \to -\infty$$

From the first equation in (3.26) we derive

$$\|D^\alpha w_i\|_E \leqslant \int_0^t \|D^\alpha \Delta_1 g_{i-2x}\|_E\, d\tau$$

The estimate for q_i is obtained in the same manner as in Lemma 3.1. Then

$$\|D^\alpha g_{ix}\|_E \leqslant \|D^\alpha w_i\|_E + \|D^\alpha q_i\|_E \leqslant N_1 t_0 + N_1 \varepsilon^{1/2} t_0^{3/4} \tag{3.27}$$

The equation and the initial condition for w_1 in (3.26) are differentiated with respect to s. Multiplying by $\partial w_1 / \partial s$ and integrating over the domain E taking into account (3.15), we find

$$\left\| D^\alpha \frac{\partial w_1}{\partial s} \right\|_E \leqslant \int_0^t \left\| D^\alpha \Delta_1 \frac{\partial g_{-1x}}{\partial s} \right\|_E d\tau \leqslant \varepsilon^{1/2} K_3 t_0$$

The estimate for q_1 is obtained in the same manner as in Lemma 3.1. Then

$$\left\| D^\alpha \frac{\partial g_{1x}}{\partial s} \right\|_E \leqslant \left\| D^\alpha \frac{\partial w_1}{\partial s} \right\|_E + \left\| D^\alpha \frac{\partial q_1}{\partial s} \right\|_E \leqslant \varepsilon^{1/2} K_4 t_0 \tag{3.28}$$

Taking into account (3.25) and also the fact that $\mathbf{g}_0 = 0$, we obtain successively the same estimates for g_{ix} and $g_{iy} (i \geqslant 2)$. The estimate for $\partial g_{iz} / \partial s$ is obtained from (3.27) and the condition $\operatorname{div} \mathbf{g}_i = 0$. For an estimate of the second derivative $\partial^2 g_{iz} / \partial s^2$ at first the equation is written out for $\partial g_{iz} / \partial s$ and a separation analogous to (3.26) is performed. The arguments are repeated which are similar to the derivation of (3.28).

T h e o r e m 3.2. Let conditions (3.14) and (3.18) be satisfied. Here j_1 and j_2 are sufficiently large numbers. Then for the solution of the problem (1.1)–(1.4) on any finite interval $(0 < t < t_0)$ for $\varepsilon \to 0$ asymptotic expansions (2.1) are applicable for which estimates of the form (3.9) are valid. In (3.9) C_l depend on functions which enter into the initial and boundary conditions of problem (1.1)–(1.4) and on their derivatives with respect to x and y.

This theorem follows directly from Theorem 3.1 and Lemmas 3.1–3.4. Lemmas 3.3 and 3.4 allow to establish the necessary estimates for functions which are determined in the pth step of the iteration process and which depend on functions determined in the two preceding steps. Lemmas 3.1 and 3.2 ensure the necessary estimates for the initial terms of the asymptotic expansion (2.1).

4. Some particular cases. 4.1°. Let us examine the plane motion of a fluid caused by initial elevation of the free surface

$$\zeta_* = \frac{Q}{\pi} \frac{b}{x^2 + b^2}, \qquad b > 0, \quad b' = \alpha b, \quad Q' = \alpha^2 Q \tag{4.1}$$

Here Q' is the area of the elevated fluid.

1°. According to Sect. 2 we obtain that $\mathbf{g}_{-1} = 0$, $\theta_{-1} = 0$. Then for determination of $\mathbf{v}_0$, p_0, ζ_0 we arrive at the problem of motion of an ideal fluid under the action of initial elevation of the free surface ζ_*.

We compute $\overline{\Phi \zeta_*}$ and using the first formula in (2.12) we obtain

$$\zeta_0 = \frac{Q}{\pi} \int_0^\infty e^{-\xi b} \cos \sqrt{\lambda \xi}\, t \cos \xi x d\xi \tag{4.2}$$

Expanding $\cos \sqrt{\lambda \xi}\, t$ in a series and integrating term by term, we find

$$\zeta_0 = \frac{Q}{\pi (x^2 + b^2)^{1/2}} \sum_{n=0}^\infty (-1)^n \frac{n!}{(2n)!} \omega_1{}^n T_{n+1}, \qquad \omega_1 = \frac{\lambda t^2}{(x^2 + b^2)^{1/2}} \tag{4.3}$$

Here and in the subsequent text $T_n = T_n \left(b / (x^2 + b^2)^{1/2} \right)$ are Chebyshev polynomials of the first kind (8.940_1 [20]).

The series (4.3) converges uniformly for any bounded ω_1, however it is inconvenient for numerical analysis in the case of large values of ω_1. For these another expression for ζ_0 is indicated.

At first we express ζ_0 from (4.2) using formula 8.953_2 of [20], in terms of parabolic cylindrical functions $D_\mu(z)$

$$\zeta_0 = \frac{Q}{\pi}\operatorname{Re}\left\{\frac{\exp{}^1/_4 z^2}{2(b-i|x|)}\,[D_{-2}(-z)+D_{-2}(z)]\right\},\quad z=\frac{i\sqrt{\lambda}t}{[2(b-i|x|)]^{1/2}} \tag{4.4}$$

Then, using relationship 3.2 (19) of [22], we derive

$$\zeta_0 = \frac{Q}{\pi}\operatorname{Re}\left\{\frac{1}{b-i|x|}\left[\frac{\sqrt{\pi}}{\sqrt{2}}ze^{z^2/2}\Phi\left(\frac{z}{\sqrt{2}}\right)+1\right]\right\},\quad \Phi(u)=\frac{2}{\sqrt{\pi}}\int_0^u e^{-\xi^2}d\xi \tag{4.5}$$

Substituting in (4.5) the probability integral $\Phi(u)$ by its asymptotic representation (8.254 of [20] in the case $\arg z \neq 0$ and pp. 116–117 of [23] in the case $\arg z = \pi/2$) and then separating the real part, we find the formula for the elevation of the free surface ζ_0 which is convenient for large values of ω_1

$$\zeta_0 = \frac{Q}{(b^2+x^2)^{1/2}}\left\{\frac{\omega_1^{1/2}}{2\sqrt{\pi}}e^{-\varkappa}\sin\left(\frac{3\varphi}{2}-\frac{\omega}{4}\right)-\frac{1}{\pi}\sum_{k=1}^{n}\frac{(2k-1)!!}{(\omega_1/2)^k}T_{k-1}+O\left(\frac{1}{\omega_1^{n+1}}\right)\right\}$$

$$\varkappa=\frac{\lambda t^2 b}{4(b^2+x^2)},\quad \omega=\frac{\lambda t^2|x|}{b^2+x^2},\quad \varphi=\operatorname{arctg}\frac{|x|}{b},\quad \omega_1\gg 1 \tag{4.6}$$

Here and in the following $O(m)=cm$, where c is a constant. It follows from (4.6) that for $\omega_1 \gg 1$ and $\varkappa \gg 1$

$$\zeta_0 = -\frac{2Q}{\pi\lambda t^2}\left[1+O\left(\frac{1}{\omega_1}\right)\right] \tag{4.7}$$

For $\omega_1 \gg 1$ and $\varkappa \ll 1$ the principal contribution to ζ_0 is made by terms of wave character.

2°. Let us find the first correction due to viscosity to the elevation of the free surface. Using (2.12) we obtain

$$\xi_2 = \frac{2Q}{\pi}\int_0^\infty \xi^2 e^{-\xi b}\left[\frac{\sin\sqrt{\lambda\xi}\,t}{\sqrt{\lambda\xi}}-t\cos\sqrt{\lambda\xi}\,t\right]\cos\xi x\,d\xi \tag{4.8}$$

From here we find various representations for ξ_2 similarly as was done in the derivation of (4.3)–(4.7)

a) Representation of ξ_2 in the form of a series

$$\xi_2 = -\frac{2Qt}{\pi(x^2+b^2)^{3/2}}\sum_{n=0}^{\infty}(-1)^n\frac{2n}{2n+1}\frac{(n+2)!\,\omega_1^n}{(2n)!}T_{n+3} \tag{4.9}$$

b) Expression of ξ_2 in terms of parabolic cylindrical functions

$$\xi_2 = -\frac{2Qt}{\pi}\operatorname{Re}\left[S_4+\frac{i}{(\lambda t^2)^{1/2}}S_5\right]$$

$$S_n = \frac{\Gamma(n)\exp{}^1/_4 z^2}{[2(b-i|x|)]^{1/2 n}}[D_{-n}(-z)+(-1)^n D_{-n}(z)] \tag{4.10}$$

c) Representation of ξ_2 by the probability integral

$$\xi_2 = -\frac{Qt}{2\pi}\operatorname{Re}\left\{\frac{1}{(b-i|x|)^3}\left[\sqrt{\frac{\pi}{2}}e^{z^2/2}\Phi\left(\frac{z}{\sqrt{2}}\right)\left(z^5+9z^3+9z-\frac{3}{z}\right)+ z^4+8z^2+3\right]\right\} \tag{4.11}$$

d) Asymptotic expansion of ξ_2 for $\omega_1 \gg 1$

$$\xi_2 = \frac{Qt}{2\pi(x^2+b^2)^{3/2}}\left\{\frac{(\pi\omega_1)^{1/2}}{2}e^{-\varkappa}\sum_{n=0}^{3}\left(\frac{\omega_1}{2}\right)^{2-n}q_{2n}\sin\left(\frac{\omega}{4}-\frac{(11-2n)\varphi}{2}\right)+\right.$$

$$\left.+\sum_{k=2}^{N}\frac{8k(k^2-1)(2k-1)!!}{(\omega_1/2)^{k+1}}T_{k-2}+O\left(\frac{1}{\omega_1^{N+2}}\right)\right\} \qquad (4.12)$$

$$q_{20}=1,\quad q_{21}=-9,\quad q_{22}=9,\quad q_{23}=3,\quad \omega_1\gg 1$$

For $\varkappa \gg 1$ it follows from here that

$$\xi_2 = \frac{576Q}{\pi\lambda^3 t^5}\left[1+O\left(\frac{1}{\omega_1}\right)\right],\qquad \omega_1\gg 1 \qquad (4.13)$$

3°. Now let us find the next correction due to viscous forces. Utilizing (2.12), we find

$$\xi_3 = \frac{2^{3/2}Qt}{\pi\lambda^{1/4}}\int_0^\infty \xi^{11/4}e^{-\xi b}\cos\xi x M(\sqrt{\xi\lambda t^2})\,d\xi + \frac{3\sqrt{2}Q}{\pi\lambda^{3/4}}\int_0^\infty \xi^{7/4}e^{-\xi b}\cos\xi x N(\sqrt{\xi\lambda t^2})\,d\xi$$

$$M(u)=\cos uC(\sqrt{u})+\sin uS(\sqrt{u}),\quad N(u)=\cos uS(\sqrt{u})-\sin uC(\sqrt{u})$$

$$C(u)=\sqrt{\frac{2}{\pi}}\int_0^u\cos\xi^2 d\xi,\qquad S(u)=\sqrt{\frac{2}{\pi}}\int_0^u\sin\xi^2 d\xi \qquad (4.14)$$

From here, using equations $8.253_{2,3}$ of [20], we derive

$$\xi_3 = \frac{16Bt^{3/2}}{\pi^{3/2}(x^2+b^2)^2}\sum_{n=0}^{\infty}(-1)^n\frac{n2^{2n}(n+3)!}{(4n+3)!!}\omega_1^n T_{n+4} \qquad (4.15)$$

The representation of ξ_3 in terms of parabolic cylindrical functions has the form

$$\xi_3 = -\frac{Qt\sqrt{2}}{\sqrt{\pi}\,\lambda^{1/4}}\mathrm{Re}\left\{\frac{e^{1/4 z^2}}{[2(b-i|x|)]^{15/4}}\left[2D_{\frac{13}{2}}\left(\frac{z}{i}\right)+\frac{3i}{z}D_{\frac{11}{2}}\left(\frac{z}{i}\right)\right]\right\}-$$

$$-\frac{Q}{\pi^{3/2}\lambda\sqrt{t}}\frac{1}{(x^2+b^2)^{3/2}}\left[8T_3-\frac{6T_2}{\omega_1}+\frac{315}{2^2\omega_1^2}T_1+O\left(\frac{1}{\omega_1^3}\right)\right],\quad \omega_1\gg 1 \quad (4.16)$$

In order to obtain formulas (4.16), in each of the integrals (4.14), a substitution of the variable of integration is made setting $a=\lambda t^2x^{-2}u$. The interval of integration is broken up into two intervals $[0, K^{-q}]$ and $[K^{-q},\infty)$, where $K=\lambda t^2|x|^{-1}$, q is a positive number and is selected on the basis of the condition that the rejected terms must be of the same order of smallness. Integrals of the finite interval are estimated; in integrals over an infinite interval the functions M and N are replaced by their integral representations according to formulas $8.256_{3,4}$ of [20]. In double integrals the trigonometric functions are expanded in Maclaurin series with remainder terms in Lagrange's form. The inner integrals are computed. The integral containing the remainder term is estimated. In one-dimensional integrals the interval of integration is extended to zero by subtracting and estimating the corresponding integrals. Then, after computing the obtained integrals and utilizing the relationship 9.248_1 of [20], we arrive at the Eq. (4.16).

From (4.16) substituting the parabolic cylindrical functions by their asymptotic expansions (8.4(1) [22]), we derive

$$\xi_3 = -\frac{Qt^{3/2}}{\sqrt{\pi}}\frac{\omega_1^3}{(x^2+b^2)^2}\left\{\frac{e^{-\varkappa}}{2^{11/2}}\sum_{n=0}^{6}(-1)^n\frac{q_{3n}}{\omega_1^n}\cos\left(\frac{\omega}{4}-(7-n)\varphi\right)+\right.$$
$$\left.+\frac{1}{\pi\omega_1^4}\left[8T_3-\frac{6T_2}{\omega_1}+\frac{315}{2^3\omega_1^2}T_1+O\left(\frac{1}{\omega_1^3}\right)\right]\right\},\quad \omega_1\gg 1,\ q_{30}=1 \quad (4.17)$$

$$q_{3n}=\frac{4^n}{n!}\left[\left(-\frac{13}{4}\right)_n\left(-\frac{11}{4}\right)_n\right]-3\frac{4^{n-1}}{(n-1)!}\left[\left(-\frac{11}{4}\right)_{n-1}\left(-\frac{9}{4}\right)_{n-1}\right]$$
$$\alpha_n=\alpha(\alpha+1)\dots(\alpha+n-1),\quad \alpha_0=1$$

In the case $\omega_1\gg 1, \varkappa\gg 1$ it follows from (4.17) that

$$\xi_3=-\frac{Q}{\pi^{3/2}\lambda\sqrt{t}}\frac{1}{(x^2+b^2)^{3/2}}\left[8T_3+O\left(\frac{1}{\omega_1}\right)\right] \quad (4.18)$$

4.1. Various representations for ξ_4 are found by the same method as for ξ_0. We have

a) Representation of ξ_4 in the form of a series

$$\xi_4=\frac{2Qt^2}{\pi(x^2+b^2)^{5/2}}\sum_{n=0}^{\infty}(-1)^n\frac{(n+4)!}{(2n)!}\frac{2n}{2n+2}\omega_1^n T_{n+5} \quad (4.19)$$

b) Expression of ξ_4 in terms of parabolic cylindrical functions

$$\xi_4=\frac{2Qt^2}{\pi}\operatorname{Re}\left\{S_{10}+\frac{2i}{(\lambda t^2)^{1/2}}S_9-\frac{2}{\lambda t^2}S_8+\frac{12}{\lambda t^2}\frac{T_4}{(x^2+b^2)^2}\right\} \quad (4.20)$$

c) Representation of ξ_4 by the probability integral

$$\xi_4=\frac{Qt^2}{8\pi}\operatorname{Re}\left\{\frac{1}{(b-i|x|)^5}\left[\frac{\sqrt{2\pi}}{2}\exp\frac{z^2}{2}\Phi\left(\frac{z}{\sqrt{2}}\right)(z^9+34z^7+\right.\right.$$
$$\left.\left.+324z^5+882z^3+315z)+z^8+33z^6+293z^4+645z^2\right]\right\} \quad (4.21)$$

d) Asymptotic expansion of ξ_4 for $\omega_1\gg 1$

$$\xi_4=\frac{Qt^2}{\pi(x^2+b^2)^{5/2}}\left\{-\frac{(\pi\omega_1)^{1/2}}{2^4}e^{-\varkappa}\sum_{n=0}^{4}q_{4n}\left(\frac{\omega_1}{2}\right)^{4-n}\sin\left(\frac{\omega}{4}-\frac{(19-2n)\varphi}{2}\right)-\right.$$
$$\left.-2\sum_{n=1}^{N}\frac{(2n-1)!!\,n(n-2)(n-3)(n-4)}{(\omega_1/2)^n}T_{|n-5|}+O\left(\frac{1}{\omega_1^{N+1}}\right)\right\},\quad \omega_1\gg 1 \quad (4.22)$$

$$q_{40}=1,\quad q_{41}=-34,\quad q_{42}=324,\quad q_{43}=-882,\quad q_{44}=315$$

e) Asymptotic expansion of ξ_4 for $\omega_1\gg 1$ and $\varkappa\gg 1$

$$\xi_4=\frac{24Q}{\pi\lambda(x^2+b^2)^2}\left[T_4+O\left(\frac{1}{\omega_1^4}\right)\right] \quad (4.23)$$

Collecting the results of calculations, we find that for the initial condition (4.1) the asymptotic expansion of the elevation of the free surface has the form

$$\zeta=\zeta_0+\varepsilon^2\xi_2+\varepsilon^3\xi_3+\varepsilon^4\xi_4+R_5,\ |R_5|\leqslant M_0(t)\,\varepsilon^5 \quad (4.24)$$

The functions ζ_0, ξ_2, ξ_3, ξ_4 which enter into (4.24) are represented by formulas (4.3), (4.9), (4.15) and (4.19) in the form of series which converge uniformly for any finite values of ω_1. From formulas (4.4), (4.10), (4.16) and (4.20), ζ_0 and ξ_i are expressed in terms of parabolic cylindrical functions. Asymptotic expansions for ζ_0 and ξ_i for $\omega_1 \gg 1$ are presented in formulas (4.6), (4.12), (4.17) and (4.22). In the case $\omega_1 \gg 1$ and $\varkappa \gg 1$ the asymptotic expansions are represented by formulas (4.7), (4.13), (4.18) and (4.23).

4.2. For comparison with an example given in [1, 16] we examine the motion which is caused by initial elevation of the free surface concentrated in a point. This motion is obtained in the transition to the limit for $b \to 0$, when the action ζ_* is given in the form (4.1) according to [24].

Letting the parameter b in formulas (4.3), (4.9), (4.15) and (4.19) approach zero, and returning to dimensional variables, we find in the examined case for the elevation of the free surface the first terms of asymptotic expansion with respect to viscosity (the indices have been omitted)

$$\xi = \frac{Qgt^2}{\pi x^2}\sum_{i=0}^{4}\left(\frac{\nu t}{x^2}\right)^{1/2 i}\eta_i, \qquad \eta_i = \sum_{n=0}^{\infty}\frac{(-1)^n (2n+1+i)!\,\omega^{2n}2^{1/2 i}}{(1/2 i)!\,(4n+2+1/2 i)\,(4n+1)!} \quad (i=0,2,4)$$

$$\eta_3 = \frac{32}{\sqrt{\pi}\,\omega}\sum_{n=1}^{\infty}(-1)^n n\frac{2^{4n}(2n+3)!}{(8n+3)!!}\omega^{2n}, \quad \eta_1 = 0, \quad \omega = \frac{gt^2}{|x|} \tag{4.25}$$

The series (4.25) converge uniformly for all finite ω. The solution can also be presented in another form

$$\xi = \frac{Q\sqrt{g}\,t}{\sqrt{2\pi}\,|x|^{3/2}}\sum_{i=0}^{4}\gamma^{1/2 i}\,\mathrm{H}_i, \quad \mathrm{H}_0 = M, \quad \mathrm{H}_2 = M\left(-\frac{1}{8}+\frac{9}{2\omega^2}\right)+N\left(\frac{9}{4\omega}+\frac{3}{\omega^3}\right)+\frac{2^{3/2}}{\sqrt{\pi}\,\omega^{3/2}}$$

$$\mathrm{H}_4 = M\left(\frac{1}{2^7}-\frac{81}{2^3\omega^2}+\frac{315}{2^3\omega^4}\right)+N\left(-\frac{17}{2^5\omega}+\frac{441}{2^3\omega^3}\right)-\frac{33}{2^{11/2}\sqrt{\pi}}\frac{1}{\omega^{3/2}}+\frac{455}{2^{7/2}\sqrt{\pi}}\frac{1}{\omega^{7/2}}, \quad \gamma = \frac{\nu t}{x^2}\omega^2 \tag{4.26}$$

$$M = M\left(\frac{\omega}{4}\right), \quad N = N\left(\frac{\omega}{4}\right) \text{ (см. 4.714)}; \quad \mathrm{H}_1 = 0, \quad \mathrm{H}_3 = \frac{2^{19/2}}{\pi\omega^{3/2}}\sum_{n=1}^{\infty}(-1)^n\frac{n\,(2n+3)!}{(8n+3)!!}(4\omega)^{2(n-1)}$$

For $\omega \gg 1$ the asymptotic representation of H_3 has the form

$$\mathrm{H}_3 = \frac{1}{2^5\omega^{1/2}}\left[\sin\frac{\omega}{4}\sum_{n=0}^{3}(-1)^n\frac{q_{32n}}{\omega^{2n}} - \cos\frac{\omega}{4}\sum_{n=0}^{2}(-1)^n\frac{q_{32n+1}}{\omega^{2n+1}} + \frac{3\cdot 2^{19/2}}{\pi\omega^5} + O\left(\frac{1}{\omega^7}\right)\right]$$

In order to obtain Eqs. (4.26), it is necessary in (4.5), (4.11), (4.15), (4.16) and (4.21) to pass to the limit for $b \to 0$ and to take advantage of the fact that in this case the probability integral is expressed by Fresnel integrals [20].

In the case of large values of ω it follows from (4.26)

$$\xi = \frac{Q\sqrt{g}\,t}{2\sqrt{\pi}\,|x|^{3/2}}\left\{\cos\left(\frac{\omega}{4}-\frac{\pi}{4}\right)\left[1-\gamma\left(\frac{1}{2^3}-\frac{9}{2\omega^2}\right)+\gamma^2\left(\frac{1}{2^7}-\frac{81}{2^3\omega^2}+\frac{315}{2^3\omega^4}\right)\right]-\right.$$

$$-\sin\left(\frac{\omega}{4}-\frac{\pi}{4}\right)\left[\gamma\left(\frac{9}{2^2\omega}+\frac{3}{\omega^3}\right)-\gamma^2\left(\frac{17}{2^5\omega}-\frac{441}{2^3\omega^3}\right)\right]+\gamma^{3/2}\sqrt{2}\,\mathrm{H}_3 - \tag{4.27}$$

$$-\frac{4}{\sqrt{\pi}\,\omega^{3/2}}\left(\left[\sum_{n=0}^{N-1}(-1)^n\frac{(4n+1)!!}{(\omega/2)^{2n}}+O\left(\frac{1}{\omega^{2N}}\right)\right]+\frac{\gamma}{\omega^2}\left[\sum_{n=1}^{N-1}(-1)^n\frac{(4n-1)!!\,8n\,(4n^2-1)}{(\omega/2)^{2n}}+\right.\right.$$

$$+ O\left(\frac{1}{\omega^{2N}}\right)\Big] + \frac{\gamma^2}{\omega^4}\Big[2\sum_{n=0}^{N-1}(-1)^n\frac{(4n+1)!!\,(4n^2-1)\,(2n-2)\,(2n-3)}{(\omega/2)^{2n}} + O\left(\frac{1}{\omega^{2N}}\right)\Big]\Big)\Big\}$$

$$\omega \gg \infty, \quad \gamma \ll 0$$

Equations (4.25)–(4.27) determine the first terms of the asymptotic expansion of the elevation of the free surface in the case of motion which was caused by a δ-shaped initial elevation.

We note that for this example, the proof for asymptotic expansions in the form in which it was carried out in Sect. 3, loses its validity because the initial perturbation has infinite energy while the estimate of errors for asymptotic expansions in Sect. 3 is given in spaces with an energy norm. Nevertheless it is evident from Eqs. (4.25)–(4.27) that the practical convergence of the asymptotic expansion for the elevation of the free surface is achieved if

$$\gamma = \frac{\nu t}{x^2}, \quad \omega^2 = \frac{\nu g^2 t^5}{x^4} \ll 1 \tag{4.28}$$

Let us compare the obtained solution (4.25)–(4.27) with known results for this example.

A. From (4.25)–(4.27) for $\nu = 0$ we obtain the representations for the elevation of the free surface of the ideal fluid in the case of motion caused by a δ-shaped initial rise [16, 25, 26]

$$\zeta_0 = \frac{Qgt^2}{\pi x^2}\sum_{n=0}^{\infty}(-1)^n\frac{(2n+1)!}{(4n+2)!}\omega^{2n}, \qquad \omega = \frac{gt^2}{|x|} \tag{4.29}$$

$$\zeta_0 = \frac{Q\sqrt{g}\,t}{\sqrt{2\pi}\,|x|^{3/2}}\left[\cos\frac{\omega}{4}C\left(\frac{\sqrt{\omega}}{2}\right) + \sin\frac{\omega}{4}S\left(\frac{\sqrt{\omega}}{2}\right)\right] \tag{4.30}$$

$$\zeta_0 = \frac{Q\sqrt{g}\,t}{2\sqrt{\pi}\,|x|^{3/2}}\left[\cos\left(\frac{\omega}{4} - \frac{\pi}{4}\right) + O(\omega^{-3/2})\right] \tag{4.31}$$

Comparing the corresponding equations for the elevation of the free surface of a viscous and an ideal fluid, we note that the presence of viscosity introduces into the elevation additional terms which change the amplitude and phase characteristics of the wave.

B. The result obtained in [1] (formula (49), Sect. 8) follows from (4.25) for $\nu t \ll x^2$ and $\omega \ll 1$

$$\zeta \sim \frac{Qgt^2}{2\pi x^2} \tag{4.32}$$

With the additional constraint $\gamma^2\omega^{3/2} \gg 1$, it follows from formula (4.27) that

$$\xi \sim \frac{Q\sqrt{g}\,t}{2\sqrt{\pi}\,|x|^{3/2}}\cos\left(\frac{\omega}{4} - \frac{\pi}{4}\right)\left[1 - \frac{\gamma}{8} + \frac{1}{2!}\left(\frac{\gamma}{8}\right)^2\right] \tag{4.33}$$

This is also in agreement with results in [1] (formula (48), Sect. 8).

C. Now let us compare the elevations of free surface of an ideal fluid for the cases where the elevations are obtained due to the action of initial disturbances with finite and infinite energies:

a) It follows from (4.6) and (4.27) that waves caused by initial disturbances with finite energy are damping waves in contrast to waves caused by δ-shaped initial rise. Furthermore, in the first case the waves have a greater period than in the second.

b) If the following relationships are valid

$$\varkappa = \frac{gt^2 b}{4(x^2 + b^2)} \ll 1, \quad \frac{b}{|x|} \ll 1, \qquad b = \text{const}$$

then the elevations which are being studied coincide with accuracy to infinitely small terms.

c) If $\varkappa \gg 1$, the deformations of the free surface are substantially different.

This indicates that the forward front and the development of the wave caused by a disturbance (4.1) with finite energy can be investigated assuming an initial δ-shaped elevation. If, however, we are interested in the decay of the wave, it is necessary to carry out additional investigations.

4.3. Let us examine plane motion of fluid induced by pressure impulses

$$p_* = \frac{A}{\pi}\frac{b}{x^2+b^2}\delta(t), \qquad A' = \rho\frac{\alpha^3}{\beta}A \tag{4.34}$$

Corresponding calculations of first terms in the asymptotic expansion for $\varepsilon \to 0$ give for the elevation of the free surface

$$\xi = -\frac{At}{\pi}\sum_{i=0}^{4}(\varepsilon^2 t)^{1/2\, i} H_i, \quad H_i = \frac{1}{(x^2+b^2)^{1+i/2}}\sum_{n=0}^{\infty}(-1)^n\frac{(n+1+i)!}{(2n+1)!}a_i\omega_1^n T_{n+2+i}$$

$$a_0 = 1, \quad a_1 = 0, \quad a_2 = -2, \quad a_3 = \frac{32}{\sqrt{\pi}}\frac{(n+1)\,2^{2n}}{(4n+5)!!}(2n+1)!, \quad a_4 = 2 \tag{4.35}$$

In analogy to the procedure used in the example 4.1, we express ξ in terms of parabolic cylindrical functions

$$\xi = \frac{A}{\pi\sqrt{\lambda}}\,\mathrm{Re}\sum_{k=0}^{4}(\varepsilon^2 t)^{1/2\,k}A_k f_k, \quad A_{0,3} = 1, \quad A_{2,4} = 2$$

$$f_k = (-1)^k i S_{3+2k} \quad (k=0,2,4), \quad f_1 = 0$$

$$f_3 = \frac{2^{3/2}\sqrt{\pi t}}{\lambda^{1/4}}\frac{\exp{}^1/_4 z^2}{[2(b-i|x|)]^{17/4}}\left[D_{15/2}(-iz) + \frac{i}{2z}D_{13/2}(-iz)\right] + \tag{4.36}$$

$$+\frac{2\sqrt{t}}{\sqrt{\pi\lambda}\,(x^2+b^2)^2}\left[\frac{2T_3}{\omega_1} - \frac{15}{2\omega_1^2}T_2 + \frac{2835}{2^4\omega_1^3}T_1 + O\left(\frac{1}{\omega_1^4}\right)\right]$$

The functions S_m are determined in (4.10) and the functions z in (4.4). From this we have for $\omega_1 \gg 1$

$$\xi = \frac{A\omega_1}{4\sqrt{\pi\lambda}\,(x^2+b^2)^{3/4}}\left\{e^{-\varkappa}\sum_{k=0}^{2}\sum_{n=0}^{2k+1}\frac{(-1)^k}{k!}\left(\frac{\gamma_1}{8}\right)^k\frac{a_{kn}}{(\omega_1/2)^n}\sin\left(\frac{\omega}{4} - \frac{(8k+5-2n)\,\varphi}{2}\right) + \right.$$

$$\left. + \frac{8}{\sqrt{\pi}\,\omega_1^{3/2}}\sum_{k=0}^{2}\frac{2^{4k}}{k!}\left(\frac{\gamma}{8}\right)^k\frac{1}{\omega_1^{2k}}\left[\sum_{n=2k+1}^{N-1}\frac{(2n-1)!!\,(-n)_{2k+1}}{(\omega_1/2)^n}T_{n-2k-1} + O\left(\frac{1}{\omega_1^N}\right)\right] + \gamma_1^{3/2}f_3\right\}$$

$$f_3 = \frac{1}{2^{3/2}\sqrt{\omega_1}}\left\{e^{-\varkappa}\left[\sum_{n=0}^{7}\frac{a_{3n}}{\omega_1^n}\cos\left(\frac{\omega}{4} + (n-8)\,\varphi\right) + O\left(\frac{1}{\omega_1^8}\right)\right] + \right.$$

$$\left. + \frac{2}{\pi\omega_1^5}\left(4T_3 - \frac{15}{\omega_1}T_2 + \frac{2835}{2^3\omega_1^2}T_1\right) + O\left(\frac{1}{\omega_1^8}\right)\right\}$$

$$a_{3n} = (-4)^n\left[\frac{1}{n!}\left(-\frac{15}{4}\right)_n\left(-\frac{13}{4}\right)_n - \frac{1}{4\,(n-1)!}\left(-\frac{13}{4}\right)_{n-1}\left(-\frac{11}{4}\right)_{n-1}\right], \quad a_{k0} = 1 \tag{4.37}$$

$$a_{01} = -1, \quad a_{11} = -15, \quad a_{12} = 45, \quad a_{13} = -15, \quad a_{21} = -45, \quad a_{22} = 630, \quad a_{23} = -3150$$

$$a_{24} = 4725, \quad a_{25} = -945, \quad \varphi = \mathrm{arctg}\,|x|\,b^{-1}, \quad \gamma_1 = \varepsilon^2\lambda^2 t^5 (x^2+b^2)^{-2}$$

In the case $\varkappa \gg 1$ and $\omega_1 \gg 1$ we obtain from (4.37)

$$\xi = \frac{A\omega_1}{2^2 \sqrt{\pi\lambda}\,(x^2+b^2)^{3/4}}\left\{\frac{2^3}{\sqrt{\pi}\omega_1^{3/2}}\sum_{k=0}^{2}\frac{b_k}{k!}\left(\frac{\gamma}{8}\right)^k\frac{1}{\omega_1^{2k}}\left[\sum_{n=2k+1}^{N-1}\frac{(2n-1)!!\,(-n)_{2k+1}}{(\omega_1/2)^n}T_{n-2k-1}+\right.\right.$$
$$\left.\left.+\,O\left(\frac{1}{\omega_1^N}\right)\right]+\frac{\gamma_1^{3/2}}{2^{3/2}\sqrt{\omega_1}}\left[\frac{2^{13/2}}{\pi\omega_1^5}\left(4T_3-\frac{15}{\omega_1}T_2+\frac{2835}{2^3\omega_1^2}T_1\right)+O\left(\frac{1}{\omega_1^8}\right)\right]\right\} \qquad (4.38)$$

4.4. Let us examine the case of motion induced by an impulse of normal stress concentrated in a point. By passing to the limit for $b \to 0$ in formulas (4.35), (4.36), by returning to dimensional variables and omitting the indices, we find

$$\xi = \frac{At}{\rho\pi x^2}\sum_{i=0}^{4}\left(\frac{\nu t}{x^2}\right)^{1/2\, i}\eta_i,\quad \eta_i=\sum_{n=0}^{\infty}(-1)^n\frac{(2n+1+i)!}{(4n+1)!}\frac{2^{1/2\, i}}{(1/2\, i)!}\,\omega^{2n} \qquad (i=0,\,2,\,4)$$

$$\eta_3=\frac{16}{\sqrt{\pi\omega}}\sum_{n=1}^{\infty}(-1)^n\,n\,\frac{2^{4n}\,(2n+3)!}{(8n+1)!!}\,\omega^{2n},\qquad \eta_1=0,\qquad \omega=\frac{gt^2}{|x|} \qquad (4.39)$$

$$\xi=\frac{A\sqrt{g}t^2}{\rho\sqrt{\pi}2^{3/2}|x|^{5/2}}\sum_{i=0}^{4}\gamma^{1/2 i}\mathrm{H}_i,\quad \gamma=\frac{\nu g^2t^5}{x^4},\quad \mathrm{H}_0=N+\frac{2}{\omega}M+\left(\frac{2}{\pi\omega}\right)^{1/2},\quad \mathrm{H}_1=0$$

$$\mathrm{H}_2=\left(-\frac{1}{8}+\frac{45}{2\omega^2}\right)N+\left(-\frac{15}{4\omega}+\frac{15}{\omega^3}\right)M-\frac{1}{\sqrt{\pi}\,2^{5/2}\omega^{1/2}}+\frac{33}{\sqrt{\pi}\,2^{1/2}\omega^{5/2}}$$

$$\mathrm{H}_4=\left(\frac{1}{2^7}-\frac{315}{2^4\omega^2}+\frac{4725}{2^3\omega^4}\right)N+\left(\frac{45}{2^6\omega}-\frac{1575}{2^3\omega^3}+\frac{945}{2^2\omega^5}\right)M+$$
$$+\frac{1}{2^{13/2}\omega^{1/2}\sqrt{\pi}}-\frac{147}{\sqrt{\pi}\,2^{5/2}\omega^{5/2}}+\frac{2895}{\sqrt{\pi}\,2^{5/2}\omega^{9/2}}$$

$$M=M\left(\frac{\omega}{4}\right),\quad N=N\left(\frac{\omega}{4}\right),\quad \mathrm{H}_3=\frac{2^{11/2}}{\pi\omega^{7/2}}\sum_{n=1}^{\infty}(-1)^n\,n\,\frac{2^{4n}\,(2n+3)!}{(8n+1)!!}\,\omega^{2n}$$

$$\mathrm{H}_3=\frac{1}{2^5\omega^{1/2}}\left[\sum_{n=0}^{7}\frac{a_{3n}}{\omega^n}\cos\left(\frac{\omega}{4}+\frac{n\pi}{2}\right)+2^{11/2}+\frac{15}{\pi\omega^6}+O\left(\frac{1}{\omega^8}\right)\right],\qquad \omega\gg 1$$

For $\omega \gg 1$ it follows

$$\xi=\frac{A\omega}{2^2\rho\sqrt{\pi}g^{1/2}|x|^{3/2}}\left\{\sum_{k=0}^{2}\sum_{n=0}^{2k+1}\frac{(-1)^k}{k!}\left(\frac{\gamma}{8}\right)^k\frac{a_{kn}}{(\omega/2)^n}\sin\left(\frac{\omega}{4}+\frac{n\pi}{2}-\frac{5\pi}{4}\right)+\right. \qquad (4.40)$$
$$+\frac{8}{\sqrt{\pi}\omega^{3/2}}\sum_{k=0}^{2}\frac{2^{4k}}{k!}\left(\frac{\gamma}{8}\right)^k\left[\sum_{m=k}^{M-1}\frac{(4m+1)!!\,(-2m-1)_{2k+1}}{(\omega/2)^{2m+1}}(-1)^{m-k}+\right.$$
$$\left.\left.+\,O\left(\frac{1}{\omega^{2M+1}}\right)\right]+\gamma^{3/2}\sqrt{2}\,\mathrm{H}_3\right\}$$

Formulas (4.39) and (4.40) determine the first terms of asymptotic expansion for the elevation of the free surface in the case where the motion was induced by an impulse of normal stress concentrated in a point.

4.5. Let us examine now the case of spatial fluid motion under the action of an initial pressure impulse and initial elevation of the free surface given in the form

$$p_* = \frac{A}{2\pi}\frac{b}{(b^2+r^2)^{3/2}}\delta(t), \quad \zeta_* = \frac{B}{2\pi}\frac{b}{(b^2+r^2)^{3/2}}, \quad r^2 = x^2+y^2,\ b>0,$$

$$A' = \rho\frac{\alpha^4}{\beta}A, \quad B' = \alpha^3 B \tag{4.41}$$

Here A' is the magnitude of the pressure impulse and B' is the volume of elevated fluid.

Corresponding calculations for the elevation of the free surface give

$$\xi = -\frac{At}{\pi}\sum_{k=0}^{4}(\varepsilon\sqrt{t})^k\xi_k^{(1)} - \frac{B}{\pi}\sum_{k=0}^{4}(\varepsilon\sqrt{t})^k\xi_k^{(2)}, \quad \xi_k^{(1)} = \sum_{n=0}^{\infty}A_k\frac{(\lambda t^2)^n}{(2n+1)!}\frac{\partial^{n+2+k}}{\partial b^{n+2+k}}\frac{1}{\sqrt{b^2+r^2}}$$

$$\xi_k^{(2)} = \sum_{n=0}^{\infty}B_k\frac{(\lambda t^2)^n}{(2n)!}\frac{\partial^{n+1+k}}{\partial b^{n+1+k}}\frac{1}{\sqrt{b^2+r^2}}, \quad A_0 = \frac{1}{2}, \quad A_1 = 0, \quad A_2 = -1$$

$$A_4 = 1, \quad B_0 = \frac{1}{2}, \quad B_1 = 0 \tag{4.42}$$

$$B_2 = -\frac{2n}{2n+1}, \quad B_4 = \frac{2n}{2n+2}, \quad A_3 = \frac{-16}{\sqrt{\pi}}\frac{(n+1)\,2^{2n}(2n+1)!}{(4n+5)!!}, \quad B_3 = \frac{-8}{\sqrt{\pi}}\frac{n2^{2n}(2n)!}{(4n+3)!!}$$

(Here and in the following text the superscript (1) indicates that the corresponding expression is obtained from the action of the initial pressure impulse, while superscript (2) indicates the effect of initial elevation of the free surface.)

It follows that in the case of actions concentrated at a point ($b\to 0$) we obtain (in dimensional variables)

$$\xi = \frac{At}{2\pi\rho r^3}\sum_{i=0}\left(\frac{\nu t}{r^2}\right)^{1/2\, i}\eta_i^{(1)} + \frac{Bgt^2}{2\pi r^3}\sum_{i=0}^{4}\left(\frac{\nu t}{r^2}\right)^{1/2\, i}\eta_i^{(2)},$$

$$\eta_i^{(1)} = \frac{2^{1/2\, i}}{\left(\frac{i}{2}\right)!}\sum_{n=0}^{\infty}(-\omega^2)^n\frac{[(2n+1+i)!!]^2}{(4n+1)!}\omega^{2n} \quad (i=0,\,2,\,4)$$

$$\eta_i^{(2)} = \frac{2^{1/2\, i}}{\left(\frac{i}{2}\right)!}\sum_{n=0}^{\infty}(-1)^n\frac{[(2n+1+i)!!]^2}{(4n+1)!\,(4n+2+i/2)} \quad (i=0,\,2,\,4)$$

$$\eta_3^{(1)} = \frac{16}{\sqrt{\pi\omega}}\sum_{n=0}^{\infty}\frac{\eta_{3n}}{(8n+1)!!}, \qquad \eta_3^{(2)} = \frac{32}{\sqrt{\pi\omega}}\sum_{n=0}^{\infty}\frac{\eta_{3n}}{(8n+3)!!} \tag{4.43}$$

$$\eta_{3n} = (-1)^n[(2n+3)!!]^2 n2^{4n}\omega^{2n}, \quad \eta_1^{(1)} = \eta_1^{(2)} = 0, \quad \omega = gt^2r^{-1}$$

The series in (4.43) converge for any fixed ω. The solution (4.43) can be represented in the form

$$\xi = -\frac{At}{\rho\sqrt{2}r^3}\omega^2\sum_{i=0}\gamma^{i/2}\mathrm{H}_i^{(1)} + \frac{Bgt^2}{\sqrt{2}r^3}\omega\sum_{i=0}^{4}\gamma^{i/2}\mathrm{H}_i^{(2)}, \qquad \gamma = \frac{\nu g^2t^5}{r^4}$$

$$\begin{gathered}
H_1^{(1)} = H_1^{(2)} = 0, \quad H_0^{(1)} = \frac{F_2}{2^7} + \frac{3F_1}{2^6\omega} - \frac{J_{1/4}J_{-1/4}}{2^4\omega^2} \\
H_2^{(1)} = -\frac{F_2}{2^{10}} - \left(\frac{21}{2^9\omega} - \frac{119}{2^7\omega^3}\right)F_1 + \frac{89}{2^8\omega^2}J_{3/4}J_{-3/4} + \left(\frac{127}{2^8\omega^2} - \frac{25}{2^5\omega^4}\right)J_{1/4}J_{-1/4} \\
H_3^{(1)} = -\frac{2^{7/2}}{\pi^{3/2}\omega^6}\sum_{n=0}^{\infty}(-1)^n\frac{n2^{4n}[(2n+3)!!]^2}{(8n+1)!!}\omega^{2n} \\
\left(H_3^{(1)} = \frac{2^{-15/2}}{\pi\omega^{3/2}}\left[\sin\left(\frac{\omega}{4} - \frac{\pi}{4}\right) + O\left(\frac{1}{\omega}\right)\right], \quad \omega \gg 1\right) \\
H_4^{(1)} = \frac{F_2}{2^{14}} + \left(\frac{55}{2^{13}\omega} - \frac{3531}{2^{10}\omega^3} + \frac{12375}{2^9\omega^5}\right)F_1 - \left(\frac{471}{2^{11}\omega^2} - \frac{13293}{2^{10}\omega^4}\right)J_{3/4}J_{-3/4} - \\
-\left(\frac{131}{2^9\omega^2} - \frac{24225}{2^{10}\omega^4} + \frac{2025}{2^7\omega^6}\right)J_{1/4}J_{-1/4}, \quad H_0^{(2)} = -\frac{F_1}{2^6} + \frac{J_{1/4}J_{-1/4}}{2^4\omega} \qquad (4.44) \\
H_2^{(2)} = \left(\frac{1}{2^9} - \frac{37}{2^7\omega^2}\right)F_1 - \left(\frac{13}{2^8\omega} + \frac{9}{2^6\omega^3}\right)J_{3/4}J_{-3/4} - \left(\frac{15}{2^8\omega} - \frac{25}{2^6\omega^3}\right)J_{1/4}J_{-1/4} \\
H_3^{(2)} = \frac{2^{9/2}}{\pi^{3/2}\omega^5}\sum_{n=0}^{\infty}(-1)^n\frac{2^{4n}n[(2n+3)!!]^2}{(8n+3)!!}\omega^{2n} \\
\left(H_3^{(2)} = \frac{2^{-13/2}}{\pi\omega^{3/2}}\left[\cos\left(\frac{\omega}{4} - \frac{\pi}{4}\right) + O\left(\frac{1}{\omega}\right)\right]\omega \gg 1\right) \\
H_4^{(2)} = \left(-\frac{1}{2^{13}} + \frac{141}{2^9\omega^2} - \frac{2973}{2^9\omega^4} + \frac{9}{2^2\omega^6}\right)F_1 + \left(\frac{21}{2^{11}\omega} - \frac{129}{2^6\omega^3} - \frac{9}{2^3\omega^5}\right)J_{3/4}J_{-3/4} + \\
+\left(\frac{11}{2^{10}\omega} - \frac{759}{2^8\omega^3} + \frac{627}{2^7\omega^5}\right)J_{1/4}J_{-1/4} + \frac{18\sqrt{2}}{\pi\omega^7}
\end{gathered}$$

Here, after the expressions for $H_3^{(1)}$ and $H_3^{(2)}$, their asymptotic representations for $\omega \gg 1$ are given in parenthesis

$$F_1 = J_{3/4}J_{1/4} - J_{-1/4}J_{-3/4}, \quad F_2 = J_{3/4}J_{-1/4} + J_{1/4}J_{-3/4}, \quad J_\mu = J_\mu(\omega/8)$$

In order to obtain formulas (4.44), we represent the expressions entering into (2.14) in the following form:

$$\begin{gathered}
\frac{4a^3\sigma}{(\sigma^2+\alpha\lambda)^2} = 2a^2\frac{d}{d\sigma}\left(-\frac{a}{\sigma^2+a\lambda}\right), \quad \left(-4a^5\frac{(3\sigma^2-a\lambda)\sigma}{(\sigma^2+a\lambda)^3} = a^2\frac{d}{d\sigma}\frac{4a^3\sigma}{(\sigma^2+a\lambda)^2}\right. \\
\frac{4a^3\lambda}{(\sigma^2+a\lambda)^2} = 2a^2\frac{d}{d\sigma}\left(\frac{\sigma}{\sigma^2+a\lambda}\right) + \frac{2}{\lambda}\left[\sigma^2\left(-\frac{a}{\sigma^2+a\lambda}\right)+a\right] - \qquad (4.45) \\
-\frac{4a^5}{(\sigma^2+a\lambda)^3} = \frac{4a^3}{\sigma} + a^2\frac{d}{d\sigma}\frac{4a^3\lambda}{(\sigma^2+\alpha\lambda)^2} + \frac{1}{\lambda}\frac{4a^3\sigma^3}{(\sigma^2+a\lambda)^2} - \frac{8}{\lambda}\sigma a^2\frac{a}{\sigma^2+a\lambda}
\end{gathered}$$

Now, according to rules of operations on transforms (13, 14], we have

$$\begin{gathered}
\xi_2^{(1)} = 2tD\xi_0^{(1)}, \quad \xi_4^{(1)} = tD\xi_2^{(1)}, \quad \xi_2^{(2)} = \frac{2B}{A\lambda}\frac{\partial^2\xi_{0A}^{(1)}}{\partial t^2} + 2tD\xi_0^{(2)}, \quad D = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} \quad (4.46) \\
\xi_4^{(2)} = \frac{2B}{\lambda\pi}\frac{\partial^4}{\partial b^4}\frac{1}{\sqrt{r^2+b^2}} + tD\xi_2^{(2)} + \frac{B}{A\lambda}\frac{\partial^2\xi_2^{(1)}}{\partial t^2} - \frac{4B}{A\lambda}\frac{1}{t}\frac{\partial\xi_2^{(1)}}{\partial t} + \frac{4B\xi_2^{(1)}}{A\lambda t^2}
\end{gathered}$$

The functions $\xi_0^{(1)}$ and $\xi_0^{(2)}$ are expressed by Bessel functions [16]. Substituting these expressions into (4.46), we find the analogous expressions for $\xi_{2,4}^{(1)}$ and $\xi_{2,4}^{(2)}$, which then leads to formula (4.44).

Asymptotic representations for $H_3^{(1)}$ and $H_3^{(2)}$ are obtained in a manner which is analogous to the procedure for obtaining asymptotic representation of function ξ_3 in the plane case (formulas (4.16) and (4.17)). The difference is only that in the spatial case the Bessel function is replaced by its asymptotic expansion with a remainder term (20].

From (4.44) for $\nu = 0$ we obtain results for the ideal fluid [16]. From formulas (4.43) and (4.44) we derive

$$\zeta = \frac{At}{2\pi\rho r^3}[1 + o(1)] + \frac{Bgt^2}{4\pi r^3}[1 + o(1)], \quad \frac{\nu t}{x^2} \ll 1, \qquad \omega \ll 1 \tag{4.47}$$

$$\zeta \sim -\frac{Agt^3E}{2^{1/2}\pi\rho r^4}\,\mathrm{si}\;\frac{\omega}{4} + \frac{Bgt^2E}{2^{5/2}\pi r^3}\cos\frac{\omega}{4}, \qquad E = 1 - \frac{\gamma}{8} + \frac{1}{2!}\left(\frac{\gamma}{8}\right)^2, \quad \gamma \ll 1$$
$$\omega \gg 1, \quad \gamma^2\omega^{5/2} \gg 1 \tag{4.48}$$

Formula (4.42) determines the first terms of the asymptotic expansion for the elevation of the free surface in the case where the external action is given in the form (4.41). Formulas (4.43), (4.44), (4.47) and (4.48) give the first terms of the solution in the case where the pressure impulse and the initial elevation of the free surface are concentrated in the origin of coordinates.

Compared to analogous formulas obtained in [15], formulas (4.43) and (4.44) differ by the presence of a term proportional to $\nu^{3/2}$ and in the form of functions with terms proportional to ν and ν^2.

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